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Weighted Polynomial Approximation on Unbounded Intervals

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Müntz-type theorems are proven for sequences of the form $\{w(t)t^{\alpha_k}\}$ defined on unbounded intervals.

The classical Bernstein problem on weighted polynomial approximation is as follows: Given a continuous function $K(t)$ on $(-\infty, \infty)$, nonvanishing thereon, such that $t^k/K(t) \to 0$, $t \to \pm\infty$, determine whether or not the sequence $\{t^k; k = 0, 1, 2,...,\}$ is fundamental in the space of continuous functions $f(t)$ such that $f(t)/K(t) \to 0$, $t \to \pm\infty$, in the uniform norm with respect to the weight $1/K(t)$; i.e., whether or not, corresponding to every such f, there exist polynomials p, making $\sup_{-\infty<t<\infty} (|f(t) - p(t)|)/K(t)$ arbitrarily small. The study of this problem was initiated by Bernstein in his well-known monograph on approximation theory [1], and has received a lot of attention since. (See Timan [2; Chap. 1], Akutovicz [3], Cheng [4], Freud [5], and references therein. See also Nachbin [6] for an abstract setting). In the fifties, motivated by Bernstein's problem, Pollard [7, 8] and de Branges [9] found necessary and sufficient conditions for a sequence of the form

$$\{w(t)\, t^k; k = 0, 1, 2,...,\} \tag{1}$$

to be fundamental in the space $C_0(R)$ of continuous functions on $(-\infty, \infty)$ that vanish at infinity. By analogy with the theorem of Müntz, it is therefore natural to consider the closure properties of sequences of the form

$$\{w(t)\, t^{\alpha_k}; k = 1, 2, 3,...\} \tag{2}$$

defined on unbounded intervals.

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Let $\{\alpha_k\}$ be a sequence of complex numbers; let $C_0{}^0(R^+)$ denote the set of functions continuous on $[0, \infty)$ that vanish at zero and infinity, and let $C_0{}^0(R^+, w)$ be the set of functions in $C_0{}^0(R^+)$ that vanish wherever $w(t)$ vanishes. For any real number ϵ, let $s(\epsilon)$ denote the series

$$\sum_{k=1}^{\infty} |\alpha_k|^{-\epsilon}. \tag{3}$$

Finally, let $P(z)$ denote the canonical product of the sequence $\{\alpha_k\}$. (For the definition of canonical product, see Boas [10, p. 18, 2.6.4].) With this notation we have

THEOREM 1. *Let $w(t)$ be a continuous function on R^+ that does not vanish identically thereon, and assume that for every $\beta > 0$ there is a number c (that depends on β), such that*

$$|w(t)| \leqslant c \exp(-|\ln t|^{\beta})$$

for all $t > 0$. Then for the sequence (2) *to be fundamental in $C_0{}^0(R^+, w)$, it is sufficient that $P(z)$ be of order larger than* 1 *(in particular that $s(\epsilon)$ be divergent for some $\epsilon > 1$). Conversely, if there is a number $\delta > 0$ such that $|\operatorname{Re}(\alpha_k)| > \delta |\alpha_k|$ for all k, then for* (2) *to be fundamental in $C_0{}^0(R^+, w)$ it is necessary that $s(1)$ be divergent.*

The closure of sequences of the form (2) in the space $L_2(0, \infty)$ has been studied by Fuchs [11] and by Boas and Pollard [12].

Applying Theorem 1 it is easy to obtain a density theorem for the whole line. For such a theorem we must, of course, assume that $\alpha_k = n_k/d_k$, where n_k is a natural number, d_k is an odd natural number, and n_k and d_k have no common factors, for under these conditions t^{α_k} will also be defined for negative values of t. We shall say that α_k is odd if n_k is odd, and that α_k is even if n_k is even. By $s_e(\epsilon)$ we shall denote the series (3), where the summation is over the set of all k for which α_k is even, and by $s_0(\epsilon)$ the series similarly defined for odd α_k. We shall use $P_e(z)$ to denote the canonical product of the even elements of the sequence $\{\alpha_k\}$, and $P_0(z)$ to denote the canonical product of the odd elements of this sequence. Finally, $C_0{}^0(R, w)$ will stand for the set of all functions continuous on the real line that vanish at zero, infinity, and wherever $w(t)$ vanishes. With this notation and the restrictions imposed on the α_k, we have

THEOREM 2. *Let $w(t)$ be a continuous function on $(-\infty, \infty)$ that does not vanish identically thereon, and assume that for every $\beta > 0$ there is a number c, such that*

$$|w(t)| \leqslant c \exp(-|\ln|t||^{\beta}) \tag{4}$$

for all $t \neq 0$. *Then for* (2) *to be fundamental in* $C_0{}^0(R, w)$, *it is sufficient that* $P_e(z)$ *and* $P_0(z)$ *be of order larger than* 1 (*in particular, it is sufficient that* $s_e(\epsilon)$ *and* $s_0(\epsilon)$ *be divergent for some* $\epsilon > 1$). *Conversely, if there is a number* $\delta > 0$ *such that* $\mathrm{Re}(\alpha_k) > \delta \mid \alpha_k \mid$ *for all* k, *then for* (2) *to be fundamental in* $C_0{}^0(R, w)$ *it is necessary that both* $s_e(1)$ *and* $s_0(1)$ *be divergent.*

One may wonder whether the sufficient conditions in Theorem 1 and in Theorem 2 are also necessary. That this is not the case is illustrated by an example given after the proof of Theorem 2. The question of whether the necessary conditions in these theorems are also sufficient is still unsettled.

Remark. Note that $w(0) = 0$; thus if $w(t) \neq 0$ elsewhere, from Theorem 1 we can at most obtain sufficient conditions for (2) to be fundamental in the space $C_0{}^0(R)$ of continuous functions that vanish at zero and infinity, and not conditions for (2) to be fundamental in $C_0(R)$.

Let $C[-1, 1]$ denote the set of functions continuous in $[-1, 1]$. With the same restrictions on the α_k as in Theorem 2, we have

THEOREM 3. *The sequence* $\{1, t^{\alpha_1}, t^{\alpha_2}, t^{\alpha_3},\ldots\}$ *is fundamental in* $C[-1, 1]$ *if and only if both the sequence* $\{1, t^{\alpha_k};\ \alpha_k$ *even*$\}$ *and the sequence* $\{1, t^{\alpha_k};\ \alpha_k$ *odd*$\}$ *are fundamental in* $C[0, 1]$.

Theorem 3 is probably well known to workers in the field, although it does not seem to have been mentioned in the literature; it clearly supplements the classical theorem of Müntz–Szász, which gives conditions for the completeness in $C[0, 1]$ (cf. Paley and Wiener [13, p. 36]). Since the proof of Theorem 3 is similar to that of Theorem 2 but much simpler, it will be omitted.

Proof of Theorem 1. *Sufficiency.* A well-known corollary of the Hahn–Banach theorem implies that the proof of the sufficiency is equivalent to showing that any continuous linear functional on $C_0{}^0(R^+, w)$ that annihilates the elements of the sequence (2) vanishes identically. Let R' be the set of points in $(-\infty, \infty)$ at which $w(t)$ does not vanish. Since $w(t)$ is continuous, R' is open and therefore locally compact, whence we can apply the Riesz representation theorem (cf. Rudin [14, p. 131]) to conclude that the problem is equivalent to showing that if μ is a (bounded) complex measure on R' such that

$$\int_{R'} w(t)\, t^{\alpha_k}\, d\mu(t) = 0; \qquad k = 1, 2, 3,\ldots, \tag{5}$$

then $\mu = 0$.

Let

$$f(z) = \int_{R'} w(t)\, t^z\, d\mu(t) = \int_{R'} w(t) \exp(z \ln t)\, d\mu(t).$$

From the hypotheses we readily see that $f(z)$ is defined for all z. Indeed, let α be any number larger than 1, and let β be such that $\alpha^{-1} + \beta^{-1} = 1$; then from the inequality of Young (cf. Beckenbach and Bellman [15; p. 15]), we see that

$$|x \ln t| \leqslant \alpha^{-1} | x |^{\alpha} + \beta^{-1} | \ln t |^{\beta}.$$

Thus, if $z = x + yi$, we have

$$|f(z)| \leqslant \int_{R'} | w(t)| \exp(x \ln t)\, d | \mu | (t)$$
$$\leqslant \left(\int_{R'} | w(t)| \exp(\beta^{-1} | \ln t |^{\beta})\, d | \mu | (t)\right) \exp(\alpha^{-1} | x |^{\alpha}).$$

Since the hypotheses imply that the preceding integral is bounded, we have shown that $f(z)$ is defined on the whole plane and of order 1 thereon. Moreover, an application of the theorems of Morera and Fubini readily shows that $f(z)$ is entire. Since (5) implies that $f(\alpha_k) = 0$, $k = 1, 2, 3, \ldots$, the properties of entire functions imply that $f(z)$ vanishes identically (cf. [9, p. 17, 2.5. 18]).

To prove that $\mu = 0$ note that, since μ is bounded, it can be represented as the difference of two measures, each having bounded and positive real and imaginary parts. The assertion now follows by making the change of variable $x = \ln t$, and applying Bochner's theorem on the uniqueness of the Fourier–Stieltjes transform (cf. Cotlar and Cignoli [16, p. 523, Theorem 3.1.9(c)]).

Necessity. Assume (2) is fundamental. Since $w(t)$ is continuous and does not vanish identically there is a closed interval $[a, b]$, $a > 0$, on which $w(x)$ does not vanish. It is thus readily seen that $\{t^{\alpha_k}; k = 1, 2, 3, \ldots\}$ is fundamental in the space of functions continuous on $[a, b]$, and the divergence of $s(1)$ follows from a generalization of the Müntz theorem due to Luxemburg and Korevaar (cf. [17, p. 30, Theorem 6.1]). Q.E.D.

Proof of Theorem 2. *Sufficiency.* Assume first that $w(t)$ is an even function. Every function in $C_0{}^0(R, w)$ can be represented as the sum of an even and an odd function. Let $g(t)$ be an even function in $C_0{}^0(R, w)$. From Theorem 1 and the hypotheses we know that there exists a sequence $\{p_n(t)\}$, in the linear span of the sequence $\{t^{\alpha_k}; \alpha_k \text{ even}\}$, such that $\{w(t)\, p_n(t)\}$ converges uniformly to $g(t)$ in R^+; however, since the functions $P_n(t)$ and $g(t)$ are even, the convergence is uniform in the whole line. Since a similar reasoning applies if $g(t)$ is odd, the conclusion follows.

In the general case, note that the set S of functions of bounded support that vanish in some open set that contains the origin and the set of points at which $w(t)$ vanishes, is dense in $C_0{}^0(R, w)$. Let therefore $g(t)$ be in S, and let $g_1(t) = | w(t)| + | w(-t)|$, $t \neq 0$, $g_1(0) = 0$. Clearly the function $g_2(t) =$

$[g_1(t)\cdot g(t)]/w(t)$, if $w(t)\neq 0$, $g_2(t)=0$, if $w(t)=0$ is in $C_0{}^0(R)$. Since $g_1(t)$ is even, we know from what was proved in the preceding paragraph that there is a sequence $\{p_n\}$, of linear combinations of the functions $\{t^{\alpha_k}; k=1,2,3,...,\}$, such that the sequence $\{g_1(t)\,p_n(t)\}$ converges to $g_2(t)$ uniformly on R. Since by hypothesis $|w(t)|\leqslant|g_1(t)|$, if $w(t)\neq 0$, we have

$$\begin{aligned}|w(t)\,p_n(t)-g(t)| &= |[w(t)]/[g_1(t)]|\,|g_1(t)\,p_n(t)-g_2(t)|\\ &\leqslant |g_1(t)\,p_n(t)-g_2(t)|,\end{aligned}$$

whereas, if $w(t)=0$, $w(t)\,p_n(t)-g(t)=0$, whence the conclusion follows.

Necessity. Assume first that $w(t)$ is even, and let (2) be fundamental in $C_0{}^0(R,w)$. Let $g(t)$ be any function in $C_0{}^0(R^+,w)$, and extend it to the whole line by stipulating that it should be even. Since $g(t)$ is even, it is clear that it is in $C_0{}^0(R,w)$. By hypothesis there is a sequence $\{p_n\}$, of linear combinations of the functions t^{α_k}, such that $\{w(t)\,p_n(t)\}$ converges to $g(t)$ uniformly on R. Since both $w(t)$ and $g(t)$ are even, it is clear that also $\{w(t)\,p_n(-t)\}$ converges to $g(t)$ uniformly on R. Adding, we thus conclude that also $\{\frac{1}{2}[p_n(t)+p_n(-t)]\,w(t)\}$ converges uniformly to $g(t)$ on R. But $\frac{1}{2}[p_n(t)+p_n(-t)]\,w(t)$ is a linear combination of elements of the sequence

$$\{w(t)\,t^{\alpha_k};\ \alpha_k \text{ even}\}. \tag{6}$$

We have therefore established that (6) is fundamental in $C_0{}^0(R^+,w)$, and the divergence of $s_e(1)$ follows from Theorem 1. A similar reasoning, involving odd functions, is used to prove the divergence of $s_0(1)$.

In the general case, assume (2) is fundamental. Let $w_1(t)=\min\{|w(t)|, |w(-t)|\}$; clearly $w_1(t)$ is an even function such that $|w_1(t)|\leqslant|w(t)|$ for all t. Hence, by considering functions of bounded support that vanish in some open set that contains the origin and the set of points at which $w_1(t)$ vanishes, as in the proof of the sufficiency, we readily see that the sequence $\{w_1(t)\,t^{\alpha_k}; k=1,2,3,...,\}$ is fundamental in $C_0{}^0(R,w_1)$. Since $w_1(t)$ is even, we have reduced the problem to the one considered in the previous paragraph, whence the conclusion follows. Q.E.D.

The following example shows that the sufficient conditions in Theorems 1 and 2 are not necessary.

EXAMPLE 1. From the theorem of [7] it is easy to see that the sequence (1), with $w(t)=\exp(-t^2)$ is fundamental in $C_0(R)$ (to prove that (3) of [7] holds, set $p_n(x)=\sum_{k=0}^{n}(x^{2k}/k!)$). It is readily seen that the sequence that is obtained from (1) by removing the term corresponding to $k=0$ is fundamental in $C_0{}^0(R)$. Let $w_1(t)=\exp(-|t^{-1}|)\,w(t)$; then $w_1(t)$ satisfies a condition of the form (4). Since $0\leqslant w_1(t)\leqslant w(t)$, by considering functions of bounded support that vanish in a neighborhood of the origin, as in the proof of the sufficiency in Theorem 2, we conclude that the sequence $\{w_1(t)\, t^k;\ k = 1, 2, 3,...\}$ is fundamental in $C_0{}^0(R)$ and therefore also in $C_0{}^0(R^+)$. However, the canonical products of the sequence of integers, of even integers, and of odd integers are all entire functions of order 1.

The reader will notice that the necessary conditions in Theorems 1 and 2 are of a global nature, for no restrictions are imposed on the weight function $w(t)$. The sufficient conditions, on the other hand, are based on the assumption that for all $\beta > 0$ either $|w(x)| \exp(-|\ln(x)|^\beta)$ is bounded (for Theorem 1) or that $|w(x)| \exp(-|\ln|x||^\beta)$ is bounded (for Theorem 2), thus leaving a gap that should be the subject of further research. At this point we should remark that if for instance $|w(x)| \exp(-|\ln(x)|^\beta)$ is not bounded for all $\beta > 0$, then the sufficiency part of Theorem 1 is not necessarily true. That this may indeed happen is shown in the following:

EXAMPLE 2. The sequence $\{\exp[-(\pi/2)|\ln t|]\, t^{1-2i\ln k};\ k = 1, 2, 3,...\}$ is not fundamental in $C_0{}^0(R^+)$. Indeed, making an obvious change of variable we see that the preceding asertion is equivalent to stating that $\{\exp[-(\pi/2)|x| + (1 - 2i \ln k)\, x];\ k = 1, 2, 3,...\}$ is not fundamental in $C_0{}^0(R)$. Applying now a theorem of Paley and Wiener [18, p. 766, Theorem 1] (this theorem also appears in [13, p. 35], but there is a misprint), we infer that the preceding sequence is not fundamental in $L_2(R)$. Since $C_0{}^0(R)$ is dense in $L_2(R)$, the conclusion follows.

Note added in proof. The fundamentality of sequences of the form (2) in $L_p(R^+)$, $1 \leq p < \infty$, has been studied by K. Endl in Der Müntzsche Satz beim Übergang vom endlichen zum unendlichen Intervall, *Acta Math. Acad. Sci. Hungar.* **22** (1971), 139–146.

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